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TURBULENCE INVARIANTS IN INCOMPRESSIBLE MAGNETOHYDRODYNAMICS

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Invariants of the form

$$\int \langle v_i(\mathbf{r}+\mathbf{R})\, v_j(\mathbf{r})\rangle R_m R_n d\mathbf{R}$$

were obtained in incompressible hydrodynamics for the case of homogeneous but anisotropic turbulence. The expression $\langle v_i(\mathbf{r}+\mathbf{R})v_j(\mathbf{r})\rangle$ is the velocity correlation function. This set of the conservation laws represents a generalization of the Loitsianskii [2] invariant for the case of arbitrary homogeneous turbulence. Analogs of these invariants are also found successfully in incompressible magnetohydrodynamics for a homogeneous anisotropic medium. The anisotropy results from the presence of a constant magnetic field H_0.

Let us consider the equation of incompressible magnetohydrodynamics, passing to new variables

$$\frac{\partial v_i}{\partial t} - (V_A\nabla)\, h_i = -\nabla_i P - \nabla_k (v_k v_i - h_k h_i) + \nu\Delta v_i$$

$$\frac{\partial h_i}{\partial t} - (V_A\nabla)\, v_i = \nabla_k (h_k v_i - v_k h_i) + \nu_m \Delta h_i$$

$$\operatorname{div}\mathbf{v} = \operatorname{div}\mathbf{h} = 0$$

$$\mathbf{h} = \frac{\mathbf{H}-\mathbf{H}_0}{(4\pi\rho)^{1/2}}, \qquad P = \frac{1}{\rho}\left(P + \frac{H^2}{8\pi}\right), \qquad \nu_m = \frac{c^2}{4\pi\sigma} \tag{1}$$

Here V_A is the Alfven velocity, ν is viscosity and ν_m is the magnetic viscosity.

In the following we shall consider the turbulence within the framework of such a system. Since in this system the waves which may propagate in the region of transparency obey the dispersion law $\omega = (\mathbf{k}\mathbf{V}_A)$, it is expedient to call such turbulence the Alfven turbulence.

The turbulence is analyzed with the help of equations which we obtain for the velocity correlation function $R_{ij}^v = \langle v_i v_j'\rangle (v_j' \equiv v_j(x'))$ and the magnetic field correlation function $R_{ij}^h = \langle h_i h_j'\rangle$. We note that in the case of homogeneous turbulence all two-point moments depend only on $r = x - x'$. Then from (1) follows

$$\frac{\partial R_{ij}^v}{\partial t} - (V_A\nabla)\,\{\langle h_i v_j'\rangle - \langle h_j' v_i\rangle\} = -\nabla_i \langle P v_j'\rangle + \nabla_j \langle P' v_i\rangle -$$
$$-\nabla_l \{\langle (v_l v_i - h_l h_i)\, v_j'\rangle - \langle (v_l' v_j' - h_l' h_j')\, v_i\rangle + 2\nu\Delta R_{ij}^v$$

$$\frac{\partial R_{ij}^h}{\partial t} - (V_A\nabla)\,\{\langle v_i h_j'\rangle - \langle v_j' h_i\rangle\} = \nabla_l \{\langle (h_l v_i - v_l h_i)\, h_j'\rangle - \tag{2}$$
$$-\langle (h_l' v_j' - v_j' h_j')\, h_i)\, h_i\rangle + 2\nu_m \Delta R_{ij}^h$$

We eliminate the moments of the form $\langle v_i h_j'\rangle$ by combining both equations of (2). Performing the Fourier transformation with respect to r on the resulting equation, we obtain

$$\frac{\partial R_{ij}^*}{\partial t} = -ik_i B_{pj} + ik_j B_{ip} + ik_l B_{i,lj} - ik_l B_{li,j} - 2k^2(\nu R_{ij}^v + \nu_m R_{ij}^h) \tag{3}$$

$$R_{ij}^* = R_{ij}^v + R_{ij}^h$$

$$\int \langle P v_j' \rangle \, e^{-i\mathbf{kr}} d\mathbf{r} = B_{pi}(\mathbf{k})$$

$$\int \{\langle (v_l v_i - h_l h_i) \, v_j' \rangle - \langle h_l v_i - v_l h_i) \, h_j \rangle\} \, e^{-i\vec{\mathbf{kr}}} d\mathbf{r} = B_{li,j}(\mathbf{k})$$

By the third and fourth equation of (1) the correlation functions of the form $\langle \ldots, v_j' \rangle$ $\langle \ldots, h_j' \rangle$ have the following property:

$$k_j \langle \ldots, v_j' \rangle_k = 0, \quad k_j \langle \ldots, h_j' \rangle = 0 \tag{4}$$

Using this property we obtain

$$k^2 B_{pj}(k) = - k_l k_i B_{li,j}(k)$$
$$k^2 B_{ip}(k) = - k_l k_j B_{i,lj}(k) \tag{5}$$

We assume the functions $R_{ij}^{v,h}(k)$, $B_{i,lj}(k)$ and $B_{pj}(k)$ have Taylor expansions in $\mathbf{k}$. We note that these correlation functions contain the scalar P, the polar vector $\mathbf{v}$ and the axial vector $\mathbf{h}$. We shall utilize this property in writing the expansions

$$R_{ij}^{v,h}(k) = f_{ij,mn}^{v,h} k_m k_n + \ldots$$

$$k_j B_{ip}(k) = b_{ij,mn} k_m k_n + \ldots$$

$$k_l B_{i,lj}(k) = \Lambda_{ij,mn} k_m k_n + \ldots$$

From (5) we can conclude that $b_{ij,mn} = - \Lambda_{ij,mn}$, i.e.

$$\frac{d}{dt}(f_{ij,mn} + f_{ij,mn}^h) = 0, \quad f_{ij,mn} = f_{ij,mn}^v + f_{ij,mn}^n = \mathrm{const}$$

These invariants written in the coordinate form

$$f_{ij,mn} = \int \{R_{ij}^v(\mathbf{r}) + R_{ij}^h(\mathbf{r})\} \, r_m r_n d\mathbf{r}$$

are analogs of the Loitsianskii invariant for Alfven turbulence.

We note that in deriving the invariants we have postulated the analyticity of the functions $R_{ij}(\mathbf{k}, t)$, $B_{i,lj}(\mathbf{k}t)$ and $B_{ip}(\mathbf{k}t)$ at $k = 0$. This means that the correlation functions $R_{ij}(\mathbf{r}, t)$, $B_{i,lj}(\mathbf{r}, t)$ and $B_{ip}(\mathbf{r}, t)$ decay exponentially as $\mathbf{r} \to \infty$. The above requirement is not always correct. The nonanalyticity of the correlation functions at $k = 0$ leads to the fact that in a number of cases (see [3]) the Loitsianskii invariants are not preserved in the nonlinear stage of development of turbulence.

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